Europ. J. Combinatorics (1998) **19**, 559–565
Article No. ej980225

The Spectra of Complementary Subgraphs in a Strongly Regular Graph

D. DE CAEN

As an application of Jacobi's identity on complementary minors, one can exhibit a simple explicit relation between the characteristic polynomials of any pair of complementary induced subgraphs in a strongly regular graph. In particular, there is an explicit relation between the spectra of the first and second subconstituents with respect to any vertex. Several consequences will be presented; for example, there follow rather strong parametric restrictions on an SRG that has some second subconstituent isomorphic to an antipodal distance-regular graph of diameter three. The latter class of graphs also admits a palatable complementary-minors formula; this will be stated, together with an application.

1. The Spectra of Complementary Subgraphs

We start by fixing some notation. Let M be any $n \times n$ matrix; one may take the indexing set for rows and columns to be $[n] = \{1, 2, \ldots, n\}$. If $T \subseteq [n]$, let $M[T]$ denote the principal submatrix of M whose row and column indices are T. (For convenience we set $M[\emptyset] = 1$.) Also put $\overline{T} = [n] \setminus T$, the complement of T; and $|M|$ will denote the determinant of M. Recall Jacobi's identity ([3, p. 21]; see also [5, p. 52], for an elegant proof): If M is invertible, then for any $T \subseteq [n]$

$$|M[\overline{T}]| = |M||M^{-1}[T]|. \tag{1}$$

One may apply this formula to the characteristic polynomials of graphs and their subgraphs. Let D be a directed graph on n vertices, with adjacency matrix A; the characteristic matrix is $C = xI_n - A$, where x is an indeterminate. By (1) we have

$$|xI_{n-t} - A[\overline{T}]| = |xI_n - A||(xI_n - A)^{-1}[T]| \tag{2}$$

where $t = |T|$. (Note that $xI_n - A$ is always invertible, when interpreted over the rational function field $\mathbb{C}(x)$.) The meaning of formula (2) is that the characteristic polynomial of $D[\overline{T}]$, the subgraph of D induced on the vertex set $\overline{T}$, is related to some other data involving the complementary vertex set T. However, note that $(xI_n - A)^{-1}[T]$ is not the same as $(xI_t - A[T])^{-1}$, so that the characteristic polynomials of $D[T]$ and $D[\overline{T}]$ are not related by a simple formula. Nevertheless, there are nontrivial applications of (2), e.g., to walk-generating functions; cf. [5, Ch. 4]. Here we will show that for strongly regular graphs, there is a version of (2) that is more useful for applications.

Our terminology will follow van Lint and Wilson [9, Ch. 21]. A *strongly regular graph* SRG (v, k, λ, μ) is a k- regular undirected graph G on v vertices, such that every pair of adjacent vertices have exactly λ common neighbours and every pair of non-adjacent vertices have exactly μ common neighbours. The adjacency matrix $A = A(G)$ satisfies, in addition to $AJ = JA = kJ$,

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J \tag{3}$$

where $I = I_v$ and $J = J_v$ is the all-ones matrix of order v. The characteristic matrix $C = xI - A$ satisfies

0195-6698/98/070559 + 07 $30.00/0

$$C^2 = x^2 I - 2xA + A^2 \tag{4}$$

and

$$(-2x + \lambda - \mu)C = -x(2x + \mu - \lambda)I + 2xA - (\lambda - \mu)A. \tag{5}$$

Adding (4) and (5) (and using (3)) we obtain

$$C[C - (2x + \mu - \lambda)I] = -q(x)I + \mu J,$$

where $q(x) = x^2 + (\mu - \lambda)x + (\mu - k)$. (We remark that the two non-principal eigenvalues of A are just the roots of $q(x)$.) Therefore

$$\begin{aligned} C^{-1} &= [-q(x)I + \mu J]^{-1}[C - (2x + \mu - \lambda)I] \\ &= \left[-\frac{1}{q(x)}I + \frac{\mu}{q(x)(-q(x) + \mu v)}J\right][-(x + \mu - \lambda)I - A] \end{aligned}$$

or

$$\begin{aligned} q(x)C^{-1} &= \left[I + \frac{\mu}{q(x) - \mu v}J\right][(x + \mu - \lambda)I + A] \\ &= A + (x + \mu - \lambda)I + \frac{\mu(x + k + \mu - \lambda)}{q(x) - \mu v}J. \end{aligned}$$

From the relation $k(k - \lambda - 1) = \mu(v - k - 1)$ (cf. [9, p. 232]), one easily finds that $q(x) - \mu v = (x - k)(x + k + \mu - \lambda)$. Hence we have the following simple form of the inverse of the characteristic matrix $C = xI - A$:

$$q(x)C^{-1} = A + (x + \mu - \lambda)I + \frac{\mu}{x - k}J. \tag{6}$$

From (6) and Jacobi's identity (2), we readily obtain the following.

THEOREM 1. *Let G be any SRG (v, k, λ, μ), with adjacency matrix A and characteristic matrix $C = xI - A$. For any subset T of vertices, with $|T| = t$,*

$$|C[\overline{T}]| = q(x)^{-t}|C| \left| A[T] + (x + \mu - \lambda)I_t + \frac{\mu}{x - k}J_t \right| \tag{7}$$

where $q(x) = x^2 + (\mu - \lambda)x + \mu - k$.

In the standard notation $q(x) = (x - r)(x - s)$, where $r > 0$ has eigenvalue multiplicity f and $s < 0$ has multiplicity g. Thus in (7) we may write $q(x)^{-t}|C|$ as $(x - k)(x - r)^{f-t}(x - s)^{g-t}$. As for the right-most determinant in (7), there is an explicit relation to the spectrum of $A[T]$ when the latter has constant row sums, i.e., when the induced subgraph $G[T]$ is regular; for then $A[T]$ commutes with J. Thus if $G[T]$ is d-regular then $A[T] + (x + \mu - \lambda)I_t + \frac{\mu}{x-k}J_t$ has top eigenvalue (i.e., constant row sum) equal to $d + (x + \mu - \lambda) + \frac{\mu t}{x-k} = [(x - k)(x + \mu - \lambda + d) + \mu t]/(x - k)$. The other eigenvalues are $x + \mu - \lambda + \omega$, as ω ranges over the eigenvalues of $A[T]$ with $\omega = d$ omitted once, i.e., the spectrum of $A[T]$ is a multiset, and $\omega = d$ will appear more than once if $G[T]$ is disconnected. We thus have the following.

COROLLARY 2. *If $G[T]$ is d-regular then the complementary induced subgraph $G[\overline{T}]$ has characteristic polynomial*

$$|C[\overline{T}]| = [(x - k)(x + \mu - \lambda + d) + \mu t]\,(x - r)^{f-t}(x - s)^{g-t}\prod_{\omega}(x + \mu - \lambda + \omega) \tag{8}$$

where ω ranges over the eigenvalues of $A[T]$ with $\omega = d$ omitted once.

Now fix any vertex p in G; $N_i = N_i(p)$ will denote the set of vertices at distance i from p $(i = 1, 2)$. The subgraphs $G[N_i]$ are called the (first and second) subconstituents with respect to p. Since $G[N_1]$ is λ-regular, on k vertices, we easily obtain from (8) a relation between the spectra of $A[N_1]$ and $A[N_2]$.

COROLLARY 3. *For any vertex p,*

$$|C[N_2(p)]| = (x-k+\mu)(x-r)^{f-k}(x-s)^{g-k}\prod_{\omega}(x+\mu-\lambda+\omega) \tag{9}$$

where ω ranges over the eigenvalues of $A[N_1(p)]$ with $\omega = \lambda$ omitted once.

EXAMPLE 4. We may use formula (9) to show the nonexistence of SRG $(28, 9, 0, 4)$. Indeed, one readily computes that $r = 1 (f = 21)$ and $s = -5$ $(g = 6)$; also since $\lambda = 0$ we see that $N_1(p)$ is the empty graph on nine vertices, for each p. Hence from (9)

$$|C[N_2(p)]| = (x-5)(x-1)^{12}(x+5)^{-3}(x+4)^8,$$

which is a proper rational function, an absurdity. On the other hand, SRG (28, 9, 0, 4) is already known not to exist, since it fails one of the Krein conditions; it is also possible to show nonexistence by elementary counting arguments (cf. [9, Problem 21D]).

EXAMPLE 5. Similarly one can prove the nonexistence of SRG $(184, 48, 2, 16)$. Here $r = 2$ $(f = 160)$ and $s = -16$ $(g = 23)$; and so in (9) for any p,

$$|C[N_2(p)]| = (x-32)(x-2)^{112}(x+16)^{-25}\prod_{\omega}(x+14+\omega).$$

Since $|C[N_2]|$ is a polynomial, this forces $\omega = 2$ with multiplicity at least 25. This is impossible: since 2 is the valency of $G[N_1]$, this means that $G[N_1]$ has at least 25 connected components, whereas a 2-regular graph on 48 vertices obviously has at most 16 components. As in Example 4, this parameter set is also excluded by one of the Krein conditions, cf. [1, p. 89].

These complementary-subgraph formulae are naturally useful when one assumes that a strongly regular graph has an induced subgraph with restricted eigenvalue structure. For example Cameron *et al.* [2] studied SRG (v, k, λ, μ)'s G with the property that for some vertex p both $G[N_1(p)]$ and $G[N_2(p)]$ are strongly regular. If $G[N_i(p)]$ has non-principal eigenvalues r_i (multiplicity f_i) and s_i (multiplicity g_i) for $i = 1, 2$, then (9) yields

$$(x-r_2)^{f_2}(x-s_2)^{g_2} = (x-r)^{f-k}(x-s)^{g-k}(x+\mu-\lambda+r_1)^{f_1}(x+\mu-\lambda+s_1)^{g_1}. \tag{10}$$

Clearly (10) severely limits the possibilities, and one can easily deduce some of the results of [2]. (Note that our Corollary 3 is a slightly more precise version of Theorem 5.1 in [2].) In the same way, one can prove some of the results of Haemers and Higman [8], who consider SRGs admitting a vertex partition into two strongly regular induced subgraphs. In the next section, we will consider a similar type of problem, namely the existence of SRG's where $G[N_2(p)]$ is distance-regular of diameter 3 for some p.

2. Distance-regular Second Subconstituent

Theorem 6 (Gardiner *et al*. [4]). *Let G be any SRG (v, k, λ, μ) that is not complete multipartite. Then for each vertex p, $G[N_2(p)]$ is connected and has diameter at most 3. If $G[N_2(p)]$ is distance-regular of diameter 3, then it is antipodal.*

We refer to Godsil [5] for an introduction to distance-regular graphs. Recall that an arbitrary graph, of diameter d, is called antipodal if the relation 'equal or at distance d' on vertices is an equivalence relation. When G is an antipodal distance-regular graph of diameter 3, then it is an r-fold covering graph of a complete graph K_n, where n is the number of antipodal classes (called *fibres*) and r is the size of each fibre. See Godsil and Hensel [7] for an in-depth study of this class of graphs (also Godsil [6] gives an up-to-date survey). In particular, they show [7, Lemma 3.1] that an antipodal distance-regular graph of diameter 3 can be specified by the triple of parameters (n, r, c_2), where n and r are as above and c_2 is the number of common neighbours to any pair of vertices at distance two. Such a graph is called an (n, r, c_2)-cover; also, following Godsil, 'an antipodal distance-regular graph of diameter three' will often just be called 'a cover'.

In view of Theorem 6, it is of interest to find (and, if possible, classify) the strongly regular graphs having some second subconstituent isomorphic to a cover. The main result of [4] is an explicit determination of those G's such that, for *every* vertex p, $G[N_2(p)]$ is a cover; they are the noncollinearity graphs of certain semipartial geometries. The authors of [4] raise the problem of finding strongly regular graphs G such that for at least one but not all p, $G[N_2(p)]$ is a cover; they give a construction showing that such graphs do exist. In what follows, we will apply (9) to derive a restriction on parameters.

To begin with, we recall [4] that if an SRG (v, k, λ, μ) G is such that, for some vertex p, $G[N_2(p)]$ is an (n, r, c_2)-cover, then the parameters of G are determined by those of $G[N_2(p)]$. Put $\gamma := c_2(c_2 - 1)/(n - c_2)$; then

$$\begin{aligned} v &= (r+1)n + c_2 + \gamma \\ k &= n - 1 + c_2 + \gamma \\ \lambda &= n - 2 - (r-1)c_2 + \gamma \\ \mu &= c_2 + \gamma. \end{aligned} \tag{11}$$

Hence γ is an integer, which gives us a restriction on the pair n, c_2. The eigenvalues of any (n, r, c_2)-cover are [7] $n - 1, -1, \theta$ and τ, where θ and τ are the two roots of the quadratic $q(x) = x^2 - (n - 2 - rc_2)x - (n - 1)$. Since $\theta\tau = -(n - 1)$, one is positive and the other negative; by convention $\theta > 0 > \tau$. The multiplicities $m(\theta)$ and $m(\tau)$ are given by

$$m(\theta) = \frac{n(r-1)\tau}{\tau - \theta};\ m(\tau) = \frac{n(r-1)\theta}{\theta - \tau}. \tag{12}$$

(Also $m(n-1) = 1$ and $m(-1) = n - 1$.) On the other hand, G has eigenvalues k and the two roots of $x^2 + (\mu - \lambda)x + (\mu - k)$, which by (11) equals $x^2 - (n - 2 - rc_2)x - (n - 1) = q(x)$; thus θ and τ are also eigenvalues of G. The G-multiplicities f, g of θ and τ can be written as

$$f = \frac{\tau(v-1) + k}{\tau - \theta};\ g = \frac{\theta(v-1) + k}{\theta - \tau}. \tag{13}$$

Collecting all this information and applying (9), we find the following result.

Theorem 7. *Let G be an SRG (v, k, λ, μ) such that, for some vertex p, $G[N_2(p)]$ is an (n, r, c_2)- cover. Then*

$$\prod_{\omega}(x + \mu - \lambda + \omega) = (x+1)^{n-1}(x-\theta)^{m(\theta)-f+k}(x-\tau)^{m(\tau)-g+k} \tag{14}$$

where θ, τ etc. are as described above, and ω ranges over the eigenvalues of $G[N_1(p)]$ with $\omega = \lambda$ omitted once. Consequently, since the left-hand side of (14) is a polynomial in x, the inequalities $m(\theta) - f + k \geq 0$ and $m(\tau) - g + k \geq 0$ hold.

These inequalities seem to be quite effective in eliminating parameter sets. For example, the following sets (n, r, c_2) with $n \leq 105$ appear to be feasible (in the sense that $\gamma = c_2(c_2 - 1)/(n - c_2)$ is an integer, etc.) but are killed by Theorem 7: (16,2,6), (33,3,9), (36,4,8), (45,3,12), (55,5,10), (56,5,12), (81, 3,21), (85,5,15), (96,3,36), (96,4,20), (105,3,27), (105,4,27),(105,7,14).

EXAMPLE 8. Do there exist covers with $(n, r, c_2) = (b^2(b + 2), b, b(b + 1))$, b some positive integer? The author suspects that they do, at least when b is a prime power, but at present they are only known to exist when $b = 1$ (a degenerate case) and $b = 2$. Note that $\gamma = c_2(c_2-1)/(n-c_2) = b(b+1)(b^2+b-1)/(b^2(b+2)-b(b+1)) = b+1$, so one might hope that such a cover arises as a second subconstituent of some strongly regular graph. The associated SRG parameters are, by (11), $v = (b+1)(b^3+2b^2+b+1)$, $k = b(b+1)(b+2)$, $\lambda = 2b^2 + 2b - 1$, $\mu = (b + 1)^2$. Now such SRGs often do exist, for example as the block-intersection graphs of $2 - (b(b + 1)^2 + 1, b + 1, 1)$ designs. This raises our hopes further. But unfortunately for such an SRG $G[N_2(p)]$ cannot be a cover. Indeed, it is straightforward to compute that $m(\tau) - g + k = -b(b^2 - 2)$, so that the second inequality in Theorem 7 is violated.

3. THE COMPLEMENTARY-MINORS FORMULA FOR COVERS

Let G be an (n, r, c_2)-cover, with adjacency matrix A. There are a total of nr vertices, partitioned into n fibres $F_1, \ldots, F_n$ each of size r. It is convenient to enumerate the vertices so that those of F_1 come first, those of F_2 second, and so on. Thus A has an easily visualized block form, with n $r \times r$ zero blocks down the main diagonal, and the off-diagonal blocks are $r \times r$ permutation matrices corresponding to the matchings between fibres. It is a simple exercise to verify the identity

$$A^2 = (n-1)I_{nr} + (a_1 - c_2)A + c_2(J_n - I_n) \otimes J_r \tag{15}$$

where $a_1 = n - 2 - (r - 1)c_2$ is the number of common neighbours to any pair of adjacent vertices, and $\otimes$ denotes the tensor product. Equation (15) is the analogue of (3) for covers; the analogue of (6) for the characteristic matrix $C = xI - A$ is

$$\begin{aligned} q(x)C^{-1} = (x + c_2 - a_1)I_{nr} + A - \frac{c_2}{x+1} I_n \otimes J_r \\ + \frac{c_2 x}{(x+1)(x-(n-1))} J_{nr} \end{aligned} \tag{16}$$

where now $q(x) = x^2 - (a_1 - c_2)x - (n-1) = (x - \theta)(x - \tau)$ in the notation of Section 2. It is tedious but routine to verify (16), using (15). The salient feature of (16) is that it expresses C^{-1} in terms of only A plus some explicit matrix. So by Jacobi's identity we obtain a fairly simple relation between the spectra of $A[T]$ and $A[\overline{T}]$, for any subset of vertices.

PROPOSITION 9. *Let A be the adjacency matrix of an (n, r, c_2)-cover. Let T be any subset of vertices, with $t = |T|$. The characteristic polynomial of A is $|C| = (x - n + 1)(x + 1)^{n-1}(x - \theta)^{m(\theta)}(x - \tau)^{m(\tau)}$. The characteristic polynomial of $A[\overline{T}]$ is*

$$\begin{aligned} |C[\overline{T}]| = (x - (n-1))(x+1)^{n-1}(x-\theta)^{m(\theta)-t}(x-\tau)^{m(\tau)-t} \\ \times \left| (x + c_2 - a_1)I_t + A[T] + \frac{c_2 x}{(x+1)(x-(n-1))} J_t - \frac{c_2}{x+1}(I_n \otimes J_r)[T] \right| \end{aligned} \tag{17}$$

where $a_1 = n - 2 - (r-1)c_2$.

We remark that if $T = T_1 \cup \cdots \cup T_n$ (partition according to fibres, i.e., $T_i = T \cap F_i$), with $t_i = |T_i|$, then $(I_n \otimes J_r)[T] = J_{t_1} \oplus \cdots \oplus J_{t_n}$ is a direct sum of J-matrices.

From (17) it is possible to relate the spectra of the first and second subconstituents with respect to any vertex p. The computation is somewhat more tedious than for the analogous formula (9); we state the result without proof.

COROLLARY 10. *Let G be an* (n, r, c_2)*-cover. The characteristic polynomial for the second subconstituent* $G[N_2(p)]$ *with respect to any vertex p is related to the spectrum of the first subconstituent* $G[N_1(p)]$ *as follows:*

$$\begin{aligned}|C[N_2(p)]| = (x-a_2)(x+c_2-a_1)^{r-2}(x-\theta)^{m(\theta)-n-r+3}(x-\tau)^{m(\tau)-n-r+3}\\ \times \prod_{\omega}(x^2+(c_2-a_1+1+\omega)x-a_1+\omega) \qquad (18)\end{aligned}$$

where $a_2 = n - 2 - c_2$ *is the valency of* $G[N_2(p)]$, *and* ω *ranges over the eigenvalues of* $G[N_1(p)]$ *with* $\omega = a_1$ *omitted once.*

In conclusion, we apply (18) to give a new proof of a theorem of Godsil and Hensel [7, Lemma 3.5].

THEOREM 11. *Let G be an* (n, r, c_2)*-cover; set* $\beta = \theta$ *or* τ. *If* β *is an integer and* $m(\beta) < n + r - 3$ *then* $\beta + 1$ *divides* c_2.

(We remark [7, Lemma 3.2] that β *is almost always an integer, so this is not a stringent hypothesis.)*

PROOF. If $m(\beta) - n - r + 3 < 0$ then in (18) $(x-\beta)$ has a negative exponent. Since $|C[N_2(p)]|$ is a polynomial in x, it follows that $(x-\beta)$ must appear somewhere else on the right-hand side of (18). It is easy to check that β cannot equal a_2 or $a_1 - c_2$; and so there exists ω in the spectrum of $A[N_1(p)]$ and a suitable β' such that $(x-\beta)(x-\beta') = x^2 + (c_2 - a_1 + 1 + \omega)x - a_1 + \omega$. One readily finds from this that $\beta' = -1 - \frac{c_2}{\beta+1}$; thus β' is rational. Hence ω is rational; and ω is also an algebraic integer (being an eigenvalue of the (0,1)-matrix $A[N_1(p)]$), so ω is an integer. Thus β' is an integer, and finally $\frac{c_2}{\beta+1}$ is an integer. □

ACKNOWLEDGEMENT

The author is supported by a grant from NSERC of Canada.

REFERENCES

1. A. E. Brouwer and J. H. van Lint, Strongly regular graphs and partial geometries, in *Enumeration and Design*, D. Jackson and S. Vanstone (eds), Academic Press, New York, 1984, pp. 85–122.
2. P. J. Cameron, J. M. Goethals and J. J. Seidel, Strongly regular graphs having strongly regular subconstituents, *J. Algebra*, **55** (1978), 257–280.
3. F. R. Gantmacher, *The Theory of Matrices*, Vol. 1, Chelsea, 1959.
4. A. D. Gardiner, C. D. Godsil, A. D. Hensel and G. F. Royle, Second neighbourhoods of strongly regular graphs, *Discrete Math.*, **103** (1992), 161–170.
5. C. D. Godsil, *Algebraic Combinatorics*, Chapman & Hall, 1993.
6. C. D. Godsil, Covers of complete graphs, in *Advanced Studies in Pure Mathematics 24, Progress in Algebraic Combinatorics, Math. Soc. Japan*, Tokyo, 1996, pp. 137–163.

7. C. D. Godsil and A. D. Hensel, Distance regular covers of the complete graph, *J. Comb. Theory, Series B*, **56** (1992), 205–238.
8. W. H. Haemers and D. G. Higman, Strongly regular graphs with strongly regular decomposition, *Linear Algebra Appl.*, **114/115** (1989), 379–398.
9. J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press, 1992.

Received 10 January 1998 and accepted 27 March 1998

D. DE CAEN
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario K7L 3N6
E-mail: `decaen@mast.queensu.ca`